Coincidence of the continuous and discrete p–adic wavelet transforms

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Abstract

We show that translations and dilations of a p–adic wavelet coincides (up to the multiplication by some root of one) with a vector from the known basis of discrete p–adic wavelets. In this sense the continuous p–adic wavelet transform coincides with the discrete p–adic wavelet transform.

The p–adic multiresolution approximation is introduced and relation with the real multiresolution approximation is described.

Relation of application of p–adic wavelets to spectral theory of p–adic pseudodifferential operators and the known results about sparsity of matrices of some real integral operators in the bases of multiresolution wavelets is discussed.

Keywords: continuous p–adic wavelet transform, discrete p–adic wavelet transform, multiresolution analysis

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1 Introduction

Wavelet analysis (see e.g. [1], [2]) is an important method used in a wide variety of applications from the theory of functions to signal analysis. For the standard wavelet analysis of functions of a real argument one can consider both the continuous wavelet transform and the discrete wavelet transform (expansion over wavelet bases).

A basis of wavelets in spaces of complex valued functions of a p–adic argument was introduced and its relation to the spectral analysis of p–adic pseudodifferential operators was investigated in [3]. One can also consider, in analogy with the real case, a continuous p–adic wavelet transform, see [4] for a discussion.

In the present paper we show that in the p–adic case, with the proper choice of a wavelet, an arbitrary translation and dilation gives (up to the multiplication by some root of one) some vector from the basis of discrete p–adic wavelets introduced in [3]. In this sense the continuous p–adic wavelet transform coincides with the discrete p–adic wavelet transform.

This result looks strange from the point of view of real analysis. The observed behavior is a manifestation of the ultrametricity of the p–adic norm. In an ultrametric space it is not possible to leave a ball making steps smaller than the diameter of this ball. This implies the

existence of locally constant functions — functions which are constant on some vicinity for any point but not necessarily constant globally.

When we apply the transformation of dilation or translation to a locally constant function of a p–adic argument, then, if the transformation is sufficiently small, the function will be invariant. This allows us to expect some properties of discreteness for continuous p–adic wavelet transform. In the present paper we show that making the right choice of wavelet it is possible to make continuous and discrete p–adic wavelet transforms to be equal.

The next construction of the present paper is the multiresolution approximation of $L^2(Q_p)$. Unlike in the real case, where translations by elements of the subgroup of integers in the real line were used, in the definition of the p–adic multiresolution approximation we use translations by elements of the factorgroup Q_p/Z_p. We show that the Bruhat–Schwartz space $\mathcal{D}(Q_p)$ of p–adic test functions can be considered as a multiresolution approximation of $L^2(Q_p)$. In this sense the construction of multiresolution analysis (MRA) is a natural property of the p–adic analysis.

These results, together with the simple relation of p–adic wavelets to spectral theory of p–adic pseudodifferential operators described in [3] (the real analogue of this relation we will discuss in Section 3), supports the point of view that the wavelet analysis is, in essence, a p–adic theory. In the real case we obtain a much more complicated wavelet analysis compared to the p–adic case.

The structure of the present paper is as follows.

In Section 2 we consider the continuous p–adic wavelet transform and show that it coincides with the discrete wavelet transform.

In Section 3 we introduce the p–adic multiresolution approximation and discuss its relation with the real multiresolution approximation. We also discuss relation of application of p–adic wavelets to spectral theory of p–adic pseudodifferential operators and the known results about sparsity of matrices of some real integral operators in the bases of multiresolution wavelets.

In Section 4, the Appendix, we put the material on p–adic analysis and p–adic wavelets, used in the present paper.

2 Continuous p–adic wavelet transform

For the notations used in the present section see the Appendix.

Let us consider the main construction of the present paper — the continuous p–adic wavelet transform generated by the p–adic wavelet of the form

$$\psi(x) = \chi(p^{-1}x)\Omega(|x|_p)$$

which is equal to the product of a character χ and the characteristic function Ω of a ball.

In the next lemma we show that any vector from the basis $\{\psi_{\gamma n j}\}$ of p–adic wavelets is a translation and dilation of the wavelet ψ, and, vice versa, any translation and dilation of the wavelet ψ belongs to the basis $\{\psi_{\gamma n j}\}$ up to multiplication by some root of one.

Equivalently, the basis of p–adic wavelets (multiplied by corresponding roots of one) can be considered as an orbit of action of the p–adic affine group in $L^2(Q_p)$ containing the function ψ.

Lemma 1 *1) Any wavelet from the basis* $\{\psi_{\gamma n j}\}$ *of* p*–adic wavelets is a translation and dilation of the wavelet*

$$\psi(x) = \chi(p^{-1}x)\Omega(|x|_p)$$

of the following form

$$\psi_{\gamma n j}(x) = \psi^{p^{-\gamma}j^{-1}, p^{-\gamma}n}(x)$$

2) The translation and dilation of the p*–adic wavelet* ψ *is proportional to a vector from the basis* $\{\psi_{\gamma n j}\}$*:*

$$\psi^{a,b}(x) = \chi\left(p^{-1}\left[(a|a|_p)^{-1} \bmod p\right](\{|a|_p b\} - [|a|_p b \bmod p])\right)\psi_{\log_p |a|_p,\, \{|a|_p b\},\, (a|a|_p)^{-1} \bmod p} \qquad (1)$$

Here $a, b \in Q_p$, $a \neq 0$.

Proof 1) One has

$$\psi_{\gamma n j}(x) = p^{-\frac{\gamma}{2}}\chi(p^{-1}j(p^{\gamma}x - n))\Omega(|p^{\gamma}x - n|_p) =$$

$$= p^{-\frac{\gamma}{2}}\chi(p^{-1}(p^{\gamma}jx - nj))\Omega(|p^{\gamma}jx - nj|_p) = p^{-\frac{\gamma}{2}}\psi(p^{\gamma}jx - nj) = \psi^{p^{-\gamma}j^{-1}, p^{-\gamma}n}(x)$$

2) Let us compute

$$\psi^{a,b}(x) = |a|_p^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right) = |a|_p^{-\frac{1}{2}}\chi\left(p^{-1}\frac{x-b}{a}\right)\Omega\left(\left|\frac{x-b}{a}\right|_p\right)$$

We have

$$a^{-1} = |a|_p\frac{a^{-1}}{|a|_p}, \qquad \Omega\left(\left|\frac{x-b}{a}\right|_p\right) = \Omega\left(||a|_p x - |a|_p b|_p\right) = \Omega\left(||a|_p x - \{|a|_p b\}|_p\right)$$

$$\chi\left(p^{-1}\frac{x-b}{a}\right)\Omega\left(\left|\frac{x-b}{a}\right|_p\right) = \chi\left(p^{-1}\frac{a^{-1}}{|a|_p}(|a|_p x - |a|_p b)\right)\Omega\left(||a|_p x - \{|a|_p b\}|_p\right) =$$

$$= \chi\left(p^{-1}\left[(a|a|_p)^{-1} \bmod p\right](|a|_p x - |a|_p b)\right)\Omega\left(||a|_p x - \{|a|_p b\}|_p\right) =$$

$$= \chi\left(p^{-1}\left[(a|a|_p)^{-1} \bmod p\right](\{|a|_p b\} - [|a|_p b \bmod p])\right)$$

$$\chi\left(p^{-1}\left[(a|a|_p)^{-1} \bmod p\right](|a|_p x - \{|a|_p b\})\right)\Omega\left(||a|_p x - \{|a|_p b\}|_p\right)$$

Taking into account the normalization, we get

$$\psi^{a,b}(x) = \chi\left(p^{-1}\left[(a|a|_p)^{-1} \bmod p\right](\{|a|_p b\} - [|a|_p b \bmod p])\right)\psi_{\log_p |a|_p,\, \{|a|_p b\},\, (a|a|_p)^{-1} \bmod p}$$

i.e. the above translation and dilation of the wavelet ψ coincides with a wavelet from the basis of p–adic wavelets times a root of one. This finishes the proof of the lemma.

The next theorem shows that, in the p–adic case, the continuous wavelet theory in essence coincides with the discrete wavelet theory.

Theorem 2 *1) The p–adic wavelet transform T, corresponding to the wavelet ψ, takes the form of an expansion over the basis of p–adic wavelets $\psi_{\gamma n j}$:*

$$(Tf)(a,b)=|a|_p^{-\frac{1}{2}}\int dx f(x)\overline{\psi}\left(\frac{x-b}{a}\right)=\langle\psi^{a,b},f\rangle=$$

$$=\chi\left(-p^{-1}\left[(a|a|_p)^{-1}\operatorname{mod}p\right](\{|a|_p b\}-[|a|_p b\operatorname{mod}p])\right)f_{\log_p|a|_p,\,\{|a|_p b\},\,(a|a|_p)^{-1}\operatorname{mod}p}$$

where $f_{\gamma n j}$ is the coefficient of expansion of the function f over the basis of wavelets.
2) For the continuous p–adic wavelet transform related to the p–adic wavelet ψ the formula for the inverse wavelet transform takes the form of the expansion of the unit operator over the basis of p–adic wavelets

$$C_\psi^{-1}\int\frac{dadb}{|a|_p^2}|\psi^{a,b}\rangle\langle\psi^{a,b}|=\sum_{\gamma\in Z,n\in Q_p/Z_p,j=1,\dots,p-1}|\psi_{\gamma n j}\rangle\langle\psi_{\gamma n j}|=1$$

Proof The first statement follows directly from Lemma 1.

Let us compute for the p–adic wavelet ψ the constant

$$C_\psi=\frac{1}{\|\psi\|^2}\int\frac{dadb}{|a|_p^2}|\langle\psi(x),\psi^{ab}(x)\rangle|^2=p^{-1}$$

Here we used (1) and the orthogonality of the wavelets in the basis, $\|\psi\|$ is the L^2–norm.

Then, using (1), we compute the integral

$$\int\frac{dadb}{|a|_p^2}|\psi^{a,b}\rangle\langle\psi^{a,b}|=\int\frac{dadb}{|a|_p^2}|\psi_{\log_p|a|_p,\,\{|a|_p b\},\,(a|a|_p)^{-1}\operatorname{mod}p}\rangle\langle\psi_{\log_p|a|_p,\,\{|a|_p b\},\,(a|a|_p)^{-1}\operatorname{mod}p}|=$$

$$=\sum_{n\in Q_p/Z_p}\int\frac{da}{|a|_p}|\psi_{\log_p|a|_p,\,n,\,(a|a|_p)^{-1}\operatorname{mod}p}\rangle\langle\psi_{\log_p|a|_p,\,n,\,(a|a|_p)^{-1}\operatorname{mod}p}|=p^{-1}\sum_{\gamma\in Z,n\in Q_p/Z_p,j=1,\dots,p-1}|\psi_{\gamma n j}\rangle\langle\psi_{\gamma n j}|$$

Therefore

$$C_\psi^{-1}\int\frac{dadb}{|a|_p^2}|\psi^{a,b}\rangle\langle\psi^{a,b}|=\sum_{\gamma\in Z,n\in Q_p/Z_p,j=1,\dots,p-1}|\psi_{\gamma n j}\rangle\langle\psi_{\gamma n j}|=1$$

This finishes the proof of the theorem.

3 Multiresolution approximation of $L^2(Q_p)$ and integral operators

The following definition of a multiresolution approximation can be found in [2].

Definition 3 *A multiresolution approximation of $L^2(R)$ is a decreasing sequence V_γ, $\gamma\in Z$, of closed linear subspaces of $L^2(R)$ with the following properties: 1)*

$$\bigcap_{-\infty}^{+\infty}V_\gamma=\{0\},\qquad\bigcup_{-\infty}^{+\infty}V_\gamma\quad\text{is dense in}\quad L^2(R);$$

2) for all $f \in L^2(R)$ *and all* $\gamma \in Z$*,*

$$f(\cdot) \in V_\gamma \Longleftrightarrow f(2\cdot) \in V_{j-1};$$

3) for all $f \in L^2(R)$ *and all* $n \in Z$*,*

$$f(\cdot) \in V_0 \Longleftrightarrow f(\cdot - n) \in V_0;$$

4) there exists a function $g \in V_0$ *such that the sequence*

$$g(\cdot - n), \quad n \in Z$$

is a Riesz basis of the space V_0*.*

It follows that there exists a function $\phi \in V_0$, such that the sequence $\phi(x-n)$, $n \in Z$ is an orthonormal basis of the space V_0. In the following we will speak only about orthonormal bases.

We denote by W_γ the orthogonal complement to V_γ in the space $V_{\gamma-1}$. Then $L^2(R)$ is the orthogonal sum of the W_γ:

$$L^2(R) = \oplus_{\gamma \in Z} W_\gamma$$

There exists a well known construction which allows to build wavelet bases from multiresolution approximations, see [1] or [2] for the details. The space W_γ possesses the basis $\{\Psi_{\gamma n}\}$, $n \in Z$ of multiresolution wavelets.

Example 1 To define a multiresolution approximation it is sufficient to fix the function ϕ. Let us take this function to be equal to the characteristic function of the interval $[0,1]$:

$$\phi(x) = \chi_{[0,1]}(x), \qquad x \in R$$

This choice of function ϕ is related to the Haar wavelets. It is easy to see that for the example under consideration all the properties of multiresolution approximation will be satisfied. Taking $\phi(x) = \chi_{[0,1]}(x)$ and $n \geq 0$ in the definition above, we obtain the multiresolution approximation for the space $L^2(R_+)$ of quadratically integrable functions on the positive half–line.

It is easy to find a p–adic analogue of the multiresolution approximation. The main difference is that in the p–adic case we should use translations by elements of the factorgroup Q_p/Z_p instead of translations by integers (we represent $n \in Q_p/Z_p$ by the numbers (4), see the Appendix).

Definition 4 *A multiresolution approximation of* $L^2(Q_p)$ *is a decreasing sequence* V_γ*,* $\gamma \in Z$*, of closed linear subspaces of* $L^2(Q_p)$ *with the following properties: 1)*

$$\bigcap_{-\infty}^{+\infty} V_\gamma = \{0\}, \qquad \bigcup_{-\infty}^{+\infty} V_\gamma \quad \text{is dense in} \quad L^2(Q_p);$$

2) for all $f \in L^2(Q_p)$ *and all* $\gamma \in Z$*,*

$$f(\cdot) \in V_\gamma \Longleftrightarrow f(p^{-1}\cdot) \in V_{\gamma-1};$$

3) for all $f \in L^2(Q_p)$ and all $n \in Q_p/Z_p$,

$$f(\cdot) \in V_0 \Longleftrightarrow f(\cdot - n) \in V_0;$$

4) there exists a function $\phi \in V_0$ such that the sequence

$$\phi(\cdot - n), \quad n \in Q_p/Z_p$$

is an orthonormal basis of the space V_0.

We do not speak here about the Riesz bases since, as we will see, in the p–adic case we have a natural example of orthonormal basis satisfying the above properties.

The space $\mathcal{D}(Q_p)$ of p–adic test functions possesses a natural filtration. Let us denote by $\mathcal{D}_\gamma(Q_p)$ the space of locally constant compactly supported functions with the diameter of local constancy p^γ:

$$|x-y|_p \le p^\gamma \Rightarrow f(x) = f(y)$$

We have the following theorem, which shows that the space $L^2(Q_p)$ possesses the natural multiresolution approximation by the space $\mathcal{D}(Q_p)$ of test functions. Therefore the multiresolution approximation can be considered as an intrinsic property of spaces of functions of a p–adic argument. Moreover, the p–adic change of variable (5) maps the multiresolution approximation of $L^2(Q_p)$ onto the multiresolution approximation of $L^2(R_+)$.

Theorem 5 *1) One has the filtration*

$$\mathcal{D}(Q_p) = \bigcup_{-\infty}^{+\infty} \mathcal{D}_\gamma(Q_p);$$

2) The sequence $\mathcal{D}_\gamma(Q_p)$, $\gamma \in Z$ is a multiresolution approximation of $L^2(Q_p)$. For this multiresolution approximation the function $\phi(x)$ can be taken to be equal to the characteristic function of the unit ball:

$$\phi(x) = \Omega(|x|_p)$$

3) For $p = 2$ the map (5) maps the multiresolution approximation $\{V_\gamma\}$ of $L^2(R_+)$ described in the Example 1 onto the multiresolution approximation $\{\mathcal{D}_\gamma(Q_p)\}$ of $L^2(Q_p)$.

The corresponding isomorphism of spaces $\mathcal{D}_\gamma(Q_p)$ and V_γ is given by the following formula:

$$\Omega(|p^\gamma x - n|_p) = \chi_{[0,p^\gamma]}(\rho(x) - p^\gamma \rho(n)), \qquad x \in Q_p, \quad n \in Q_p/Z_p.$$

Proof The first statement of the theorem means that for any locally constant function with compact support there exists a nonzero infimum of diameters of local constancy. This follows from the compactness of the support.

The proof of the second statements of the theorem is straightforward. The function ϕ in the Definition 4 can be taken equal to be to the characteristic function of the ball:

$$\phi(x) = \Omega(|x|_p)$$

The translations $\Omega(|x-n|_p)$, $n \in Q_p/Z_p$ given by (4), provide an orthonormal basis in $\mathcal{D}_0(Q_p)$. Dilations by the degrees of p give the spaces $\mathcal{D}_\gamma(Q_p)$.

The third statement follows from Lemma 7 and the above choice of the function ϕ. The space $\mathcal{D}_\gamma(Q_p)$ possesses the basis $\{\Omega(|p^\gamma x - n|_p)\}$, $x \in Q_p$, $n \in Q_p/Z_p$. The set of functions $\{\chi_{[0,p^\gamma]}(x - p^\gamma m)\}$, $m \in N_0$, $x \in R_+$ constitutes a basis in V_γ. Since the map ρ is a one-to-one correspondence between Q_p/Z_p and N_0, this finishes the proof of the theorem.

Note that, contrary to the real case, where characteristic functions and Haar wavelets are discontinuous, the multiresolution approximation of $L^2(Q_p)$ considered in the above theorem consists of spaces of continuous functions. After the p–adic change of variable the construction which was not regular (was discontinuous) in the real case, becomes highly regular in the p–adic case.

Discuss now application of wavelets to investigation of integral operators. It was found [5] that for some families of integral operators, say the Calderon–Zygmund and the pseudodifferential operators, matrices of these operators will be close to diagonal in wavelet bases for which wavelets has many momenta equal to zero. Consider the multiresolution approximation for $L^2(R)$ for which the corresponding wavelet has M vanishing moments:

$$\int \psi(x) x^m dx = 0, \qquad m = 0, 1, \dots M - 1.$$

Then [5] we have the following property of decay for matrix elements of some integral operators

$$|\alpha_{il}^\gamma| + |\beta_{il}^\gamma| + |\gamma_{il}^\gamma| \leq \frac{C_M}{1 + |i - l|^{M+1}}$$

for all

$$|i - l| \geq 2M$$

Here the matrix elements α_{il}^γ, β_{il}^γ, γ_{il}^γ are taken in the wavelet bases in the spaces V_γ and W_γ of the multiresolution approximation, the index γ is related to the scale of the wavelets and the indices i, l related to space localization (i.e. these are translation indices) of the wavelets.

This means that matrices of the corresponding integral operators in the wavelet bases are sparse and close to diagonal matrices. This result has a variety of applications, for example to numerical algorithms of fast multiplication of integral operators in wavelet bases.

Discuss now the p–adic analogue of this result. It is known [3], see the Theorem 6 in the Appendix, that p–adic pseudodifferential operators are diagonal in the basis of p–adic wavelets. Moreover, in the p–adic case we do not need any conditions of vanishing of higher moments for the wavelets. Therefore in the p–adic case we have much stronger results about the relation of wavelet bases and integral operators. The diagonality of matrices of integral operators which was approximate for the real case becomes exact for the p–adic case, and, moreover, we are able to compute the spectra of the corresponding operators.

4 Appendix

Let us recall some constructions of p–adic analysis, see [6] for details and, e.g. [7], [8] for applications. The field Q_p of p–adic numbers is the completion of Q with respect to the

p–adic norm $|\cdot|_p$, defined as follows. For any rational number we consider the representation

$$x = p^{\gamma}\frac{m}{n}$$

where p is a prime, γ is an integer, p, m, n are mutually prime numbers, $n \neq 0$. The p–adic norm is defined as follows

$$|x|_p = p^{-\gamma}$$

A p–adic number can be uniquely represented by the series

$$x = \sum_{j=\gamma}^{\infty} x_j p^j, \qquad x_j = 0, \dots, p-1$$

which converges in the p–adic norm.

A complex valued character $\chi : Q_p \to C$ of a p–adic argument (where x has the form of the above series) is defined by

$$\chi(x) = \exp\left(2\pi i \sum_{j=\gamma}^{-1} x_j p^j\right), \qquad \chi(x+y) = \chi(x)\chi(y)$$

When γ above is nonnegative the sum above is equal to zero and the character is equal to one.

The character χ is a locally constant function. The function f is locally constant if for any x there exists ε for which $\forall y$: $|x-y|_p \le \varepsilon$ we have $f(x) = f(y)$.

Another example of a locally constant function is characteristic function of a p–adic ball. We denote $\Omega(x)$ the characteristic function of the interval $[0,1]$. The characteristic function of the p–adic ball of radius 1 with center in 0 has the form:

$$\Omega(|x|_p) = \left\{ \begin{array}{ll} 1, & |x|_p \le 1 \ , \\ 0, & |x|_p > 1 \ . \end{array} \right.$$

Let us note that, unlike in the real case, the characteristic function of a p–adic ball is continuous (the same holds for an arbitrary locally constant function).

The Bruhat–Schwartz space $\mathcal{D}(Q_p)$ of p–adic test functions is the linear space of locally constant complex valued functions with compact support. Any function in $\mathcal{D}(Q_p)$ is a (finite) linear combination of characteristic functions of balls.

The Vladimirov operator of p–adic fractional differentiation has the following form

$$D^{\alpha} f(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{Q_p} \frac{f(x) - f(y)}{|x-y|_p^{1+\alpha}} d\mu(y), \qquad \alpha > 0$$

Here μ is the Haar measure on p–adic field (for which the measure of a ball is equal to the diameter of this ball). The p–adic gamma function has the form

$$\Gamma_p(-\alpha) = \frac{p^{\alpha} - 1}{1 - p^{-1-\alpha}}$$

The domain of D^{α} contains the space $\mathcal{D}(Q_p)$.

The Vladimirov operator generates a natural random walk on the p–adic field considered in [7]. For more general considerations of p–adic dynamics see [8].

The basis of discrete p–adic wavelets and its relations to the spectral theory of p–adic pseudodifferential operators is described in the following theorem [3]:

Theorem 6 *1) The set of functions* $\{\psi_{\gamma nj}\}$*:*

$$\psi_{\gamma nj}(x)=p^{-\frac{\gamma}{2}}\chi(p^{\gamma-1}j(x-p^{-\gamma}n))\Omega(|p^{\gamma}x-n|_p),$$
$$x\in Q_p,\quad \gamma\in\mathbf{Z},\quad n\in Q_p/Z_p,\quad j=1,\dots,p-1 \qquad (2)$$

is an orthonormal basis in $L^2(Q_p)$*.*
2) This basis consists of eigenvectors of the Vladimirov operator D^{α}*:*

$$D^{\alpha}\psi_{\gamma nj}=p^{\alpha(1-\gamma)}\psi_{\gamma nj} \qquad (3)$$

here $n\in Q_p/Z_p$ *in (2) are represented by the numbers*

$$n=\sum_{l=\beta}^{-1}n_l p^l,\qquad n_l=0,\dots,p-1 \qquad (4)$$

It is possible to compute, using the basis of p–adic wavelets, spectra of a very wide class of p–adic pseudodifferential operators [9]. Multidimensional p–adic wavelets in relation to spectral theory of p–adic pseudodifferential operators were discussed in [10]. Wavelets on some family of abelian locally compact groups were considered in [11]. There exists a generalization of theory of p–adic wavelets and pseudodifferential operators onto general locally compact ultrametric spaces [12], [13], [14].

There exists a natural map of p–adic numbers onto the positive half–line $R_+=[0,\infty)$, which, for $p=2$, maps the basis of 2–adic wavelets onto the basis of Haar wavelets on the half–line. This exhibits a natural relation between real and p–adic wavelet theories.

Let us discuss this relation. The wavelet basis in $L^2(R)$ contains the functions

$$\Psi_{\gamma n}(x)=2^{-\frac{\gamma}{2}}\Psi(2^{-\gamma}x-n),\quad \gamma\in Z,\quad n\in Z$$

where Ψ is some integrable mean zero function. The simplest example is the Haar wavelet

$$\Psi(x)=\chi_{[0,\frac{1}{2}]}(x)-\chi_{[\frac{1}{2},1]}(x)$$

equal to the difference of two characteristic functions.

Define the p–adic change of variable (or the Monna map) as follows:

$$\rho: Q_p\to R_+$$
$$\rho:\sum_{i=\gamma}^{\infty}x_i p^i\mapsto\sum_{i=\gamma}^{\infty}x_i p^{-i-1},\quad x_i=0,\dots,p-1,\quad \gamma\in Z \qquad (5)$$

This map is one-to-one almost everywhere and conserves the measure:

$$\mu(S)=L(\rho(S))$$

where S is a measurable set, μ is the p–adic Haar measure, L is the Lebesgue measure on the half–line. The map ρ satisfies the Hölder inequality:

$$|\rho(x)-\rho(y)|\le|x-y|_p$$

The p–adic change of variable is a one–to–one map between the group Q_p/Z_p and the set N_0 of natural numbers including zero. Here the elements of Q_p/Z_p are given by the rational numbers (4).

The following lemma and theorem were proven in [3].

Lemma 7 *For $n \in Q_p/Z_p$ and $m, k \in Z$, $k \geq m$ the map ρ satisfies the conditions*

$$\rho : p^m n + p^k Z_p \to p^{-m}\rho(n) + [0, p^{-k}] \tag{6}$$

$$\rho : Q_p \backslash \{p^m n + p^k Z_p\} \to R_+ \backslash \{p^{-m}\rho(n) + (0, p^{-k})\}, \qquad n \neq 0 \tag{7}$$

$$\rho : Q_p \backslash \{p^k Z_p\} \to R_+ \backslash [0, p^{-k})$$

Theorem 8 *For $p = 2$ the map (5) provides a one to one correspondence between the Haar basis in $L^2(R_+)$ and the basis of p–adic wavelets in $L^2(Q_p)$:*

$$\Psi_{\gamma\rho(n)}(\rho(x)) = \psi_{\gamma n 1}(x) \tag{8}$$

We understand formula (8) in the L^2 sense (since the map ρ is not a one-to-one map on the set of measure zero).

One can consider the continuous wavelet transform on the field of p–adic numbers [4]. The affine group on the field of p–adic numbers in the unitary representation G in $L^2(Q_p)$ acts by translations and dilations

$$G(a, b)f(x) = f^{a,b}(x) = |a|_p^{-\frac{1}{2}} f\left(\frac{x - b}{a}\right)$$

where $a, b \in Q_p$ and $a \neq 0$.

Assume we have a wavelet on Q_p — i.e. a mean zero complex valued function Ψ in $L^1(Q_p) \bigcap L^2(Q_p)$. The wavelet transform Tf of the function $f \in L^2(Q_p)$ is defined by

$$(Tf)(a, b) = |a|_p^{-\frac{1}{2}} \int dx f(x) \overline{\Psi}\left(\frac{x - b}{a}\right) = \langle \Psi^{a,b}, f \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(Q_p)$.

The inverse wavelet transform has the form

$$f(x) = C_\Psi^{-1} \int \frac{da db}{|a|_p^2} \langle \Psi^{a,b}, f \rangle \Psi^{a,b}(x) = C_\Psi^{-1} \int \frac{da db}{|a|_p^2} (Tf)(a, b) \Psi^{a,b}(x) \tag{9}$$

Here

$$C_\Psi = \frac{1}{\|\Psi\|^2} \int \frac{da db}{|a|_p^2} |\langle \Psi(x), \Psi^{a,b}(x) \rangle|^2$$

and $\|\Psi\|$ is the L^2–norm.

The formula for the inverse wavelet transform (9) has the equivalent form of an expansion of the unit

$$C_\Psi^{-1} \int \frac{da db}{|a|_p^2} |\Psi^{a,b}\rangle\langle\Psi^{a,b}| = 1$$

Here we use the standard notations from quantum mechanics: $|\Psi^{a,b}\rangle$ denotes the element of the Hilbert space $L^2(Q_p)$ given by the function $\Psi^{a,b}$, $\langle\Psi^{a,b}|$ is the canonically conjugate linear bounded functional on $L^2(Q_p)$, and $|\Psi^{a,b}\rangle\langle\Psi^{a,b}|$ is the orthogonal rank one projection onto $\Psi^{a,b} \in L^2(Q_p)$.

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